Viscous Incompressible Non-Newtonian Flow Around Fluid Sphere at Intermediate Reynolds Number

YOSHISUKE NAKANO and CHI TIEN

Syracuse University, Syracuse, New York

Approximate expressions of velocity profile and drag coefficients have been obtained for viscous incompressible non-Newtonian flow over a fluid sphere in the intermediate Reynolds number range.

The equations of motion of both external and internal fluids are satisfied by using Galerkin's method. The internal fluid is assumed to be Newtonian, while the external is assumed to be non-Newtonian and can be described by the power law model. Furthermore, the condition that the tangential stress is transmitted across the fluid-fluid interface without diminution is also satisfied with Galerkin's condition.

Comparisons are also made between the predicted results and the experimental data available in the literature. The possible reasons for the discrepancy are also discussed.

The study of the motion of a fluid sphere through another fluid medium has been a subject of considerable interest ever since the pioneer work of Hadamard (*6*) and Rybczynski (*14*) for the case of creeping flow. Subsequent investigators (*4, 7, 9, 10, 11, 15*) have extended the study to various flow regimes as well as to other complicating factors such as particle deformation.

All the references aforementioned considered that both the particle fluid and field fluid are Newtonian, even though it has been known for some time that certain liquid systems involved in aeration operation exhibit non-Newtonian behavior (*1, 5*).

Attempts have been made only in recent years to extend the study to non-Newtonian systems. Astarita (*2*) obtained expressions for drag coefficient of gas bubbles flowing through a non-Newtonian fluid for both creeping flow and higher Reynolds numbers based on semiquantitative arguments. His expressions, however, are applicable to the case of gas bubble flowing through power law fluid and, therefore, may not be applicable to liquid drops, especially if the viscosity of the drop liquid is of the same order of magnitude of the field fluid. Furthermore, no information is provided on the flow behavior both within and without the fluid particle, which is needed for the study of the mass transfer problem.

In a more recent study, Nakano and Tien (*12*) presented an analytical solution on the creeping flow of a Newtonian fluid sphere through a power law fluid. The solution was obtained based on a technique which is a combination of variational principle and Galerkin's method.* Expressions for external and internal stream functions are given as well as drag coefficient. For the drag coefficient, it is given as

$$C = \frac{Y}{2^n N_{Re_0}} \tag{1}$$

where Y is a function of flow behavior index, n, and a parameter, X, which can be characterized as viscosity ratio.

The purpose of the present investigation is to present approximate solutions of the flow of power law fluid over Newtonian fluid sphere at intermediate Reynolds number as an extension of the earlier work on creeping flow. The equation of motion for both external and internal fluid were satisfied by Galerkin's method. The assumption of flow at intermediate Reynolds number necessitates the retention of inertial terms for both equations and, therefore, makes the use of variational principle impractical. Kawaguti (*8*) first used the Galerkin's method for the solution of flow over rigid sphere, and his approach has been adapted to the case of fluid sphere (*7, 11*). According to these investigations, the results based on this method should be applicable for the range of $5 < N_{Re} < 40$.

The choice of the power law model for the characterization of the non-Newtonian behavior of the exterior fluid may also deserve some comment. The use of power law model is essentially a practical one, mainly because of two reasons. First, some of the liquid systems which are used in aeration processes and exhibit non-Newtonian behavior such as activated sludge were found to obey this model (*1*); second, the power law model gives the simplest mathematical model representation of non-Newtonian behavior. The use of any other model in this work would yield mathematical problems of such staggering magnitude that any kind of solution would be unlikely. Of course, the validity of this choice can only be determined by the agreement of predicted and experimental results as well as the degrees of success of correlating experimental data based on this model.

ANALYSIS

The equation of continuity and motion for this system can be written in the following tensorial form:

Yoshisuke Nakano is with the U. S. Army Cold Regions Research and Engineering Laboratory, Hanover, New Hampshire.

* A good account on the use of Galerkin's method can be found in text entitled "Approximate Method of Higher Analysis" by L. V. Kantorovich and V. I. Krylov, p. 298, Interscience, New York.

$$\frac{\partial \rho}{\partial t} = -(\rho v^j)_{,j} \tag{2}$$

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right) = -p^{,i} - \tau^{ij}_{,j} + \rho f^i \tag{3}$$

For the problem to be considered in this investigation, the following assumptions will be made: axisymmetric and steady state flow, incompressible internal and external fluids, negligible temperature and concentration gradients so that the properties of fluids are considered as constant, perfectly spherical the fluid particle, and the field exterior fluid is assumed to obey the Ostwald-de-Wael model while interior fluid can be considered as Newtonian.

The relationships between the stress tensor and the rate of deformation tensor for both external and internal fluids are given as

$$\tau^i_j = -K(\Delta_k^m \Delta_k^m)^{\frac{n-1}{2}} \Delta^i_j \quad \text{for external fluid} \tag{4}$$

$$\tau^i_j = -\mu \Delta^i_j \quad \text{for internal fluid} \tag{5}$$

By substituting Equations (4) and (5) into Equation (3) and by eliminating the pressure gradient term, the dimensionless equation of motions in terms of stream functions expressed in spherical coordinates (r, θ) for both internal and external fluids can be written as

$$f_1(r,\theta) \equiv N_{Re_i}\left[\frac{\partial \psi_i}{\partial r}\frac{\partial}{\partial \theta}\left[\frac{D^2\psi_i}{r^2 \sin^2\theta}\right] - \frac{\partial \psi_i}{\partial \theta}\frac{\partial}{\partial r}\left[\frac{D^2\psi_i}{r^2\sin^2\theta}\right]\right]\sin\theta + D^4\psi_i = 0$$

$$\text{for } 0 < r < 1 \text{ (internal fluid)} \tag{6}$$

$$f_2(r,\theta) \equiv N_{Re_o}\left[\frac{\partial \psi_o}{\partial r}\frac{\partial}{\partial \theta}\left[\frac{D^2\psi_o}{r^2\sin^2\theta}\right] - \frac{\partial \psi_o}{\partial \theta}\frac{\partial}{\partial r}\left[\frac{D^2\psi_o}{r^2\sin^2\theta}\right]\right]\sin\theta + D^4\psi_o + \left[\lambda_1\left[\frac{\partial^2\phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] + \lambda_2\frac{\partial^2\phi}{\partial r\partial\theta} + \lambda_3\frac{\partial\phi}{\partial r}\lambda_4\frac{\partial\phi}{\partial\theta}\right]\sin\theta = 0 \tag{7}$$

$$\text{for } 1 < r \text{ (internal fluid)}$$

The dimensionless velocity component v (v_r or v_θ) is defined as V/V_∞, when V is the physical velocity component and V_∞ is the relative velocity between the particle and the field fluid. ϕ and the coefficients λ_1, λ_2, λ_3, and λ_4 are given as

$$\phi = \left\{2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta\right)^2\right] + \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{4}\frac{\partial v_r}{\partial\theta}\right]^2\right\}^{\frac{n-1}{2}} \tag{8}$$

$$\lambda_1 = \frac{1}{\sin\theta}\left[\frac{\partial^2\psi_o}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\psi_o}{\partial\theta^2} - \frac{2}{r}\frac{\partial\psi_o}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi_o}{\partial\theta}\right] \tag{9a}$$

$$\lambda_2 = \frac{2}{r^2\sin\theta}\left[2\frac{\partial^2\psi_o}{\partial r\partial\theta} - \frac{3}{r}\frac{\partial\psi_o}{\partial\theta} - \cot\theta\frac{\partial\psi_o}{\partial r}\right] \tag{9b}$$

$$\lambda_3 = \frac{2}{\sin\theta}\left[\frac{1}{r^2}\frac{\partial^3\psi_o}{\partial r\partial\theta^2} + \frac{\partial^3\psi_o}{\partial r^3} - \frac{2}{r^3}\frac{\partial^2\psi_o}{\partial\theta^2} - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial^2\psi_o}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi_o}{\partial r} + \frac{2\cos\theta}{r^3\sin\theta}\frac{\partial\psi_o}{\partial\theta} - \frac{1}{r}\frac{\partial^2\psi_o}{\partial r^2}\right] \tag{9c}$$

$$\lambda_4 = \frac{1}{\sin\theta}\left[\frac{2}{r^4}\frac{\partial^3\psi_o}{\partial\theta^3} + \frac{2}{r^2}\frac{\partial^3\psi_o}{\partial r^2\partial\theta} - \frac{3\cos\theta}{r^4\sin\theta}\frac{\partial^2\psi_o}{\partial\theta^2} - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial^2\psi_o}{\partial r^2} - \frac{4}{r^3}\frac{\partial^2\psi_o}{\partial r\partial\theta} + \frac{2\cos\theta}{r^3\sin\theta}\frac{\partial\psi_o}{\partial r} + \frac{3+5\sin^2\theta}{r^4\sin^2\theta}\frac{\partial\psi_o}{\partial\theta}\right] \tag{9d}$$

The boundary conditions are

$$(v_r)_i = (v_r)_o = 0 \quad \text{at } r = 1 \tag{10a}$$

$$(v_\theta)_i = (v_\theta)_o \quad \text{at } r = 1 \tag{10b}$$

$$(\tau_{r\theta})_i = (\tau_{r\theta})_o \quad \text{at } r = 1 \tag{10c}$$

$$[(v_r)_o]^2 + [(v_\theta)_o]^2 \to 1 \quad \text{at } r \to \infty \tag{10d}$$

$$(v_\theta)_i,\ (v_r)_i \quad \text{remain finite as } r \to 0 \tag{10e}$$

Equations (6) to (10) give a complete description of the physical problem, and its solution would yield expressions of velocity profile, both inside and outside the particle.

For the approximate solution of this problem, the stream functions ψ_i and ψ_o are assumed to be

$$\psi_i = (C_1r^2 + C_2r^3 + C_3r^4)(1 - Z^2) + (D_1r^2 + D_2r^3 + D_3r^4)Z(1 - Z^2) \tag{11}$$

$$\psi_o = \left(-\frac{1}{2}r^2 + \frac{A_1}{r} + \frac{A_2}{r^2} + \frac{A_3}{r^3}\right)(1 - Z^2) + \left(\frac{B_1}{r} + \frac{B_2}{r^2} + \frac{B_3}{r^3}\right)Z(1 - Z^2) \tag{12}$$

where

$$Z = \cos\theta$$

It can be shown that ψ_o and ψ_i satisfy Equations (10d) and (10c), respectively. From Equation (10a), one has

$$A_1 + A_2 + A_3 = \frac{1}{2} \tag{13}$$

$$C_1 + C_2 + C_3 = 0 \tag{14}$$

$$B_1 + B_2 + B_3 = 0 \tag{15}$$

$$D_1 + D_2 + D_3 = 0 \tag{16}$$

From Equation (12), one has

$$A_1 + 2A_2 + 3A_3 + 1 = 2C_1 - 3C_2 - 4C_3 \tag{17}$$

$$B_1 + 2B_2 + 3B_3 = 2D_1 - 3D_2 - 4D_3 \tag{18}$$

applying the Galerkin integration for Equations (6), (7), and (10c) with r, r^2, $1/r$, and $1/r^2$ as weighting functions for the internal and external fluids, respectively:

$$\int_0^1\int_{-1}^1 r(1 - Z^2)f_1(r\cdot Z)\,dZdr = 0 \tag{19}$$

$$\int_0^1\int_{-1}^1 r^2Z(1 - Z^2)f_1(r\cdot Z)\,dZdr = 0 \tag{20}$$

$$\int_1^\infty\int_{-1}^1 \frac{1}{r}(1 - Z^2)f_2(r\cdot Z)\,dZdr = 0 \tag{21}$$

$$\int_1^\infty\int_{-1}^1 \frac{1}{r^2}(1 - Z^2)Z f_2(r\cdot Z)\,dZdr = 0 \tag{22}$$

$$\int_{-1}^{1}[(\tau_{r\theta})_i - (\tau_{r\theta})_o]_{r=1}\,(1-Z^2)dZ = 0 \quad (23)$$

$$\int_{-1}^{1}[(\tau_{r\theta})_i - (\tau_{r\theta})_o]_{r=1}\,Z(1-Z^2)dZ = 0 \quad (24)$$

From Equations (19) and (20), one has

$$N_{Re_i}\left[-8C_1D_1 + 6C_1D_3 - 5C_2D_1 + \frac{4}{3}C_2D_2 + \frac{31}{4}C_2D_3 - \frac{20}{3}C_3D_1 + \frac{36}{5}C_3D_3\right] - 14C_2 = 0 \quad (25)$$

$$N_{Re_i}\left[2C_1C_2 + \frac{3}{2}C_2^2 + \frac{6}{5}C_2C_3 - \frac{1}{2}D_1D_3 + \frac{9}{5}D_2D_3 + 2D_3^2 - 2D_1^2 - 2D_1D_2\right] + 18\left(\frac{D_3}{3} - D_1\right) = 0 \quad (26)$$

In order to circumvent the difficulties of evaluating Equations (21) to (24) caused by the presence of the non-Newtonian term ϕ and its derivatives, ϕ may be linearized approximately in the following way. First, the external velocity components were obtained from Equation (12). According to the definition of ϕ, the non-Newtonian term is found to be

$$\phi^{\frac{2}{n-1}} = \frac{2}{r^4}\,[\beta_4Z^4 + \beta_3Z^3 + \beta_2Z^2 + \beta_1Z + \beta_0\,]$$

$$= \frac{2}{r^4}\,g_0\,(r\cdot Z) \quad (27)$$

where β_i's* are functions of r and the coefficients $A_1 - A_3$, $B_1 - B_3$ of the external stream function. If ϕ is written as a Taylor series in terms of the variable $r^{(n-1)}$ at $r = 1$, one has

$$\phi_0 \simeq \phi_1\left(\frac{1}{r^{1-n}} - 1\right) + \frac{1}{2}\phi_2\left(\frac{1}{r^{1-n}} - 1\right)^2 \quad (28)$$

where

$$\phi_0 = \phi_{r=1} = [2g_0(1, Z] \quad (29)$$

$$\phi_1 = \left.\frac{\partial\phi}{\partial r^{n-1}}\right|_{r=1} \quad (30)$$

ϕ_2 is determined so that ϕ satisfies the following conditions:

$$\phi \to 1 \quad \text{as} \quad n \to 1 \quad (31a)$$

$$\phi \to 0 \quad \text{as} \quad r \to \infty \quad (31b)$$

Then

$$\phi_2 = 2(\phi_1 - \phi_0) \quad (31c)$$

assuming that $1 - n \geqq 0$. Or, in other words, the external fluid is pseudoplastic. ϕ can be written as

$$\phi = (2\phi_0 - \phi_1)\,\frac{1}{r^{1-n}} + (\phi_1 - \phi_0)\,\frac{1}{r^{2(1-n)}} \quad (32)$$

Differentiating Equation (27) with respect to r^{n-1}, one gets

$$\phi_1 = \left.\frac{\partial\phi}{\partial r^{n-1}}\right|_{r=1} = [2g_0(1, Z)]^{\frac{n-3}{2}}\,[-4g_0(1, Z) - g_1(1, Z)] \quad (33)$$

and

$$g_1(r, A) = \beta_4'Z^4 + \beta_3'Z^3 + \beta_2'Z^2 + B_1'Z + \beta_0' \quad (34)$$

$$\beta_i' = \frac{\partial\beta_i}{\partial r^{-1}}$$

After some rather lengthy calculations, one gets

$$g_0(1, Z) = \overline{\beta}_4Z^4 + \overline{\beta}_3Z^3 + \overline{\beta}_2Z^2 + \overline{\beta}_1Z + \overline{\beta}_0{}^* \quad (35a)$$

$$g_1(1, Z) = \overline{\beta}_4'Z^4 + \overline{\beta}_3'Z^3 + \overline{\beta}_2'Z^2 + \overline{\beta}_1'Z + \overline{\beta}_0'{}^* \quad (35b)$$

$\frac{\partial\phi_i}{\partial\theta}$ and $\frac{\partial^2\phi_i}{\partial\theta^2}$ are given as

$$\frac{\partial\phi_0}{\partial\theta} = (n-1)\,[2g_0(1, Z)]^{\frac{n-3}{2}}\,g_0'(1, Z)^* \quad (36)$$

$$\frac{\partial^2\phi_0}{\partial\theta^2} = (n-1)\,[2g_0(1, Z)]^{\frac{n-5}{2}}\,[(n-3)g_0^2(1, Z) + 2g_0(1, Z)g_0''(1, Z)]^* \quad (37)$$

$$\frac{\partial\phi_1}{\partial\theta} = (2g_0)^{\frac{n-5}{2}}\,[(4-4n)g_0g_0' - (n-3)g_0'g_1 - 2g_0g_1']^* \quad (38)$$

$$\frac{\partial^2\phi_1}{\partial\theta^2} = (2g_0)^{\frac{n-7}{2}}\,[4(1-n)(n-3)g_0g_0'^2 - (n-5)(n-3)g_0'^2g_1 - 4(n-3)g_0g_0'g_1' + 8(1-n)g_0^2g_0'' - 2(n-3)g_0g_0''g_1 - 4g_0^2g_1'']^* \quad (39)$$

As a further simplification, ϕ_i, $\frac{\partial\phi_i}{\partial\theta}$, and $\frac{\partial^2\phi_i}{\partial\theta^2}$ may be approximated by Fourier series as

$$\phi_i = \frac{\alpha_{io}}{2} + \sum_{j=1}^{\infty}\alpha_{ij}\cos j\theta \quad (40a)$$

$$\frac{\partial\theta_i}{\partial\theta} = \frac{\gamma_{io}}{2} + \sum_{j=1}^{\infty}\gamma_{ij}\cos j\theta \quad (40b)$$

$$\frac{\partial^2\phi_i}{\partial\theta^2} = \frac{\delta_{io}}{2} + \sum_{j=1}^{\infty}\delta_{ij}\cos j\theta \quad (40c)$$

where

$$\xi_{in} = \frac{2}{\pi}\int_0^1 x_i\cos n\theta\,dz \quad (41)$$

ξ stands for α, γ, or δ and x_i for ϕ_i, $\frac{\partial\phi_i}{\partial\theta}$, and $\frac{\partial^2\phi_i}{\partial\theta^2}$, respectively.

The above development yields expression of ϕ (and its derivatives) in a convenient form which can be explicitly integrated as required by Equations (21) to (24). If only a finite number of terms (say five) is used in the series expression of Equation (40a), the integration of Equation (21) gives

$$-\frac{8}{105}N_{Re_o}\left[2B_1 + 2B_3 + \frac{128}{7}A_1B_1 - \frac{80}{3}A_1B_3 + 46A_2B_1 + \frac{64}{3}A_2B_2 - 12A_2B_3 + \frac{784}{9}A_3B_1 + 56A_3B_2 + 16A_3B_3\right] + \sum_{k=1}^{3}i_kA_k + \sum_{k=4}^{6}i_kB_{k-3} = 0 \quad (42)$$

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* Because of the limitation of space, detailed expressions of β_i's are not presented here. Their definitions can be found in reference 13.

* Definitions of $\overline{\beta}_i$'s, $\overline{\beta}'$s, $g_0'(1, Z)$, $g_0''(1, Z)$, $g_1'(1, Z)$, and $g_1''(1, Z)$ are given in reference 13.

Similarly, from Equation (22), one has

$$\frac{32}{105} N_{Reo} \Big[-2A_2 - 5A_3 + \frac{8}{3} A_1A_2 + 7A_1A_3 + \frac{12}{5} A_2^2 + \frac{94}{11} A_2A_3 + \frac{35}{6} A_3^2 - \frac{1}{6} B_1^2 - \frac{1}{5} B_1B_3 - \frac{8}{27} B_1B_2 + \frac{4}{11} B_2B_3 + \frac{1}{2} B_3^2 \Big] + \sum_{k=1}^{3} j_kA_k + \sum_{k=4}^{6} j_kB_{k-3} = 0 \quad (43)$$

where A_k and B_k are coefficients of external stream functions [Equation (12)]. The coefficients i_k and j_k are complicated functions of the expansion coefficients of ϕ_i, $\frac{\partial \phi_i}{\partial \theta}$, and $\frac{\partial^2 \phi_i}{\partial \theta^2}$ and are given in reference *13*.

From Equations (23) and (24), one has,

$$s(3A_1 + 6A_2 + 10A_3) + t(5B_1 + 8B_2 + 12B_3) = C_2 + 2C_3 \quad (44)$$

where

$$s = -\frac{1}{48X} (48\,\bar{\alpha}_{00} + 8\,\bar{\alpha}_{02} + 3\,\bar{\alpha}_{04})^* \quad (45a)$$

$$t = \frac{1}{48X} (8\,\bar{\alpha}_{01} + 3\,\bar{\alpha}_{03})^* \quad (45b)$$

$$X = \frac{\mu_i}{K} \left(\frac{a}{v_\infty} \right)^{n-1} \quad (45c)$$

and

$$l(3A_1 + 6A_2 + 10A_3) + m(5B_1 + 8B_2 + 12B_3) = 2D_1 + 3D_2 + 5D_3 \quad (46)$$

where

$$l = \frac{1}{80X} (80\,\bar{\alpha}_{01} + 30\,\bar{\alpha}_{03})^* \quad (47a)$$

$$m = \frac{1}{80X} (80\,\bar{\alpha}_{00} + 30\,\bar{\alpha}_{02} + 15\,\bar{\alpha}_{04})^* \quad (47b)$$

The problem now is reduced to the simultaneous solution of a set of algebraic solutions. For the twelve unknowns ($A_1 - A_3$, $B_1 - B_3$, $C_1 - C_3$, and $D_1 - D_3$) there are twelve equations [Equations (13) to (18), (25), (26), (42) to (44), and (46)]. All the coefficients in these equations (for example, l, m, s, t, i_k, j_k, etc.) are given explicitly in terms of the coefficients of stream functions. Therefore, one can, at least in principle, solve this set of equations and obtain numerical values for these coefficients for given values of N_{Rei}, N_{Reo}, X, and n. Accordingly, one can calculate the drag coefficients (friction and form). They are given as

$$C_{\text{friction}} = -\frac{4}{N_{Reo}} \int_0^\pi \phi_0 \, \tau^+{}_{r\theta} \Big|_{r=1} \sin^2\theta \, d\theta = -\frac{16}{N_{Reo}} \Big[(3 + 6A_2 + 14A_3) \left(\frac{1}{3} \bar{\alpha}_{00} + \frac{1}{15} \bar{\alpha}_{02} + \frac{1}{35} \bar{\alpha}_{04} \right) + (6B_2 + 14B_3) \left(\frac{1}{15} \bar{\alpha}_{01} + \frac{1}{35} \bar{\alpha}_{03} \right) \Big] \quad (48)$$

and

$$C_{\text{form}} = -2 \int_0^\pi p^* \sin^2\theta \, d\theta - \frac{4}{N_{Reo}} \int_0^\pi \phi_0 \, \tau^+{}_{rr} \Big|_{r=1} \sin\theta \cos\theta \, d\theta = -\frac{1}{2} [(1 + A_1 + 2A_2 + 3A_3) \sin\theta + (B_1 + 2B_2 + 3B_3) \sin\theta \cos\theta]^2 + \frac{1}{N_{Reo}} \int_0^\theta \phi_0 [-(16A_2 + 50A_3) \sin\theta + (12B_1 - 30B_3) \sin\theta \cos\theta] d\theta + \frac{1}{N_{Reo}} \int_0^\theta \left[\tau^+{}_{r\theta} \frac{\partial \phi}{\partial \theta} + \tau^+{}_{\theta\theta} \frac{\partial \phi}{\partial \theta} \right]_{r=1} d\theta$$

$$= \frac{16}{15} [1 + A_1 + 2A_2 + 3A_3][B_1 + 2B_2 + 3B_3] - \frac{8}{N_{Reo}} \Big[(16A_2 + 50A_3) \left(\frac{\bar{\alpha}_{00}}{3} + \frac{\bar{\alpha}_{02}}{15} + \frac{\bar{\alpha}_{04}}{35} \right) - (12B_1 - 30B_3) \left(\frac{\bar{\alpha}_{01}}{15} + \frac{\bar{\alpha}_{03}}{35} \right) \Big] + \frac{8(n-1)}{N_{Reo}} \Big[(1 + 4A_1 + 10A_2 + 18A_3) \left(\frac{\bar{\alpha}_{10}}{3} + \frac{\bar{\alpha}_{12}}{15} + \frac{\bar{\alpha}_{14}}{35} \right) + (4B_1 + 10B_2 + 18B_3) \left(\frac{\bar{\alpha}_{11}}{15} + \frac{\bar{\alpha}_{13}}{35} \right) \Big] + \frac{32}{N_{Reo}} \Big[(1 + A_1 + 2A_2 + 3A_3) \left(\frac{\bar{\alpha}_{00}}{3} + \frac{\bar{\alpha}_{02}}{5} + \frac{\bar{\alpha}_{04}}{7} \right) + (B_1 + 2B_2 + 3B_3) \left(\frac{2}{15} \bar{\alpha}_{01} + \frac{4}{35} \bar{\alpha}_{03} \right) \Big] - \frac{4\pi}{N_{Reo}} \Big[(1 + A_1 + 2A_2 + 3A_3) \left(\frac{\bar{\gamma}_{01}}{8} + \frac{\bar{\gamma}_{03}}{16} \right) + (B_1 + 2B_2 + 3B_3) \left(-\frac{\bar{\gamma}_{00}}{4} + \frac{\bar{\gamma}_{04}}{64} \right) \Big] \quad (49)$$

RESULTS AND DISCUSSION

As stated before, the coefficient of the stream functions can be obtained from the simultaneous solution of the twelve equations. The three parameters involved are the external Reynolds number N_{Reo}, the viscosity ratio X, and the internal Reynolds number. As demonstrated in an earlier study (*11*), the internal Reynolds number has negligible effect on the external flow pattern (that is, external stream function and drag coefficient). Consequently all the numerical computations in the present investigation were made under the condition of $N_{Rei} = 0$.

The major difficulty involved in the computation work is the presence of the nonlinear term in the pertinent equations. Although these terms are related to the coefficients of the stream functions, the complexities of these relationships are so great that these equations cannot be solved directly. Instead, an iteration procedure was developed. For a given value of the flow behavior index, $n = n_o$, the nonlinear terms l, s, m, t, i_k, and j_k were first estimated by using the values of A_i, B_i, C_i, and D_i for the slightly less non-Newtonian case (that is, $n = n_o + \Delta n$). The approximate values of the coefficients of the stream functions were then obtained from the simultaneous solution of the twelve pertinent equations. Once these coefficients were known, a new set of values of l, s, m, t, i_k, and j_k could be estimated, and the same procedure was repeated until the desired accuracy was reached.

The actual computation started with slightly non-New-

* Definitions of $\bar{\alpha}_{0i}$'s are given in reference *13*.

TABLE 1. NUMERICAL VALUES OF DRAG COEFFICIENTS

$X = 0.01$

N_{Re_0}	n	$C_{fric.}$	C_{form}
	1.00	2.664(−2)	2.708
	0.90	2.675(−2)	2.885
5.0	0.80	2.684(−2)	3.059
	0.70	2.692(−2)	3.242
	0.60	2.700(−2)	3.447
	1.00	1.789(−2)	1.844
	0.90	1.796(−2)	1.956
7.5	0.80	1.801(−2)	2.060
	0.70	1.804(−2)	2.161
	0.60	1.806(−2)	2.262
	1.00	1.356(−2)	1.424
	0.90	1.361(−2)	1.504
10.0	0.80	1.365(−2)	1.576
	0.70	1.367(−2)	1.642
	0.60	1.367(−2)	1.705
	1.00	6.179(−3)	0.7773
	0.90	6.289(−3)	0.8156
25.0	0.80	6.417(−3)	0.8523
	0.70	6.579(−3)	0.8894
	0.60	6.820(−3)	0.9330

$X = 0.1$

N_{Re_0}	n	$C_{fric.}$	C_{form}
	1.00	0.2513	2.670
	0.90	0.2537	2.878
5.0	0.80	0.2567	3.130
	0.70	0.2608	3.459
	0.60	0.2671	3.925
	1.00	0.1687	1.819
	0.90	0.1704	1.963
7.5	0.80	0.1725	2.138
	0.70	0.1754	2.356
	0.60	0.1796	2.648
	1.00	0.1277	1.401
	0.90	0.1291	1.517
10.0	0.80	0.1309	1.653
	0.70	0.1332	1.818
	0.60	0.1364	2.029
	1.00	0.05769	0.7582
	0.90	0.05956	0.8376
25.0	0.80	0.06229	0.9346
	0.70	0.06721	1.069
	0.60		

TABLE 1. CONTINUED

$X = 1.00$

N_{Re_0}	n	$C_{fric.}$	C_{form}
	1.00	1.605	2.431
	0.90	1.621	2.620
5.0	0.80	1.644	2.825
	0.70	1.682	3.064
	0.60	1.748	3.370
	1.00	1.074	1.646
	0.90	1.085	1.771
7.5	0.80	1.101	1.905
	0.70	1.126	2.056
	0.60	1.167	2.229
	1.00	0.8095	1.261
	0.90	0.8182	1.355
10.0	0.80	0.8306	1.452
	0.70	0.8489	1.556
	0.60	0.8767	1.659
	1.00	0.3449	0.6349
	0.90	0.3517	0.6763
25.0	0.80	0.3606	0.7124
	0.70	0.3729	0.7471
	0.60	0.3994	0.8366

$X = 2.0$

N_{Re_0}	n	$C_{fric.}$	C_{form}
	1.00	2.290	2.310
	0.90	2.290	2.488
5.0	0.80	2.302	2.680
	0.70	2.343	2.915
	0.60	2.452	3.236
	1.00	1.530	1.560
	0.90	1.530	1.674
7.5	0.80	1.537	1.791
	0.70	1.559	1.920
	0.60	1.606	2.035
	1.00	1.152	1.193
	0.90	1.151	1.272
10.0	0.80	1.155	1.347
	0.70	1.165	1.415
	0.60	1.177	1.409
	1.00	0.4804	0.5806
	0.90	0.4784	0.5886
25.0	0.80	0.4725	0.5735
	0.70	0.4596	0.5265
	0.60	0.4398	0.4804

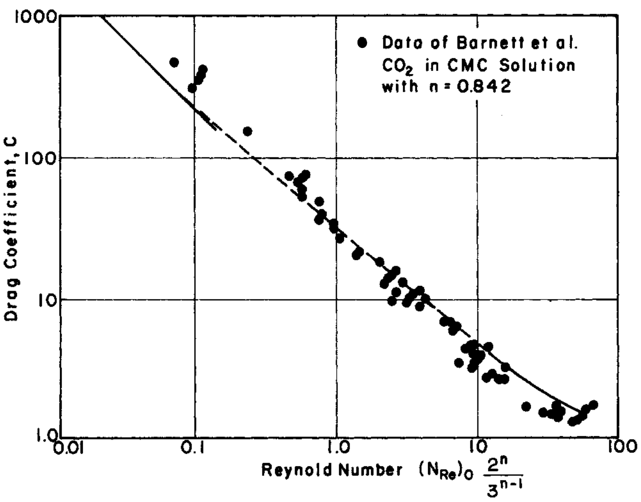

Fig. 1. Comparison with experimental data (3).

tonian case, $n = 0.99$, with the value of l, s, m, t, etc., first estimated from the Newtonian case which has been studied previously (*11*). Succeeding computations were made with decreasing values for the flow behavior index by using $\Delta n = 0.01$.

The ranges of parameters covered are as follows:

$$0.01 < X < 2.0$$

$$5 < N_{Re_0} < 25$$

$$0.6 < n < 1.0$$

Numerical values of the coefficients of the stream functions as well as drag coefficients for various cases are summarized in Tables 1 and 2.* The relaionship of drag

* Table 2 has been deposited as document 00928 with the ASIS National Auxiliary Publications Service, c/o CCM Information Sciences, Inc., 22 W. 34th St., New York 10001 and may be obtained for $2.00 for microfiche or $5.00 for photocopies.

coefficient vs. external Reynolds number is also shown in Figure 1. Also included in Figure 1 are the results for creeping flow regime (*12*). As pointed out by Hamielec and co-workers (*7*), the drag coefficient at intermediate Reynolds number for the Newtonian case cannot be joined smoothly with the solution of Hadamard-Rybczynski; the same trend was found true for power law fluid. This is because the creeping flow solution is valid for $N_{Re} << 1$, and the present analysis is correct for $5 < N_{Re} < 40$. The two solutions were joined together by drawing a straight line between $N_{Re} = 0.1$ and $N_{Re} = 5$. This is also shown in Figure 1.

Comparison of the results of this analysis with available experimental data have been made and are shown in Figure 1. In Figure 1, the experimental results of Barnett, Humphrey, and Litt (*2*) were used. The experimental values of the drag coefficient were determined based on observation of the rise of a single carbon dioxide bubble through aqueous solutions of carbonymethyl cellulose. Some scattering was observed between the experimental results and those based on the present investigation. Although the exact reasons of this discrepancy are unknown, there are some major differences between the experimental conditions and the basic assumptions used in the present analysis. First, the carbon dioxide bubble used in the experimental work was not spherical and could be better approximated by spherical ellipsoid. Second, and probably more important, the results were taken in connection with mass transfer studies. The size of the bubble decreased continually owing to mass transfer, as it rose to the extent of more than 60% of its original value. In view of these differences, the agreement appears to be reasonably good.

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NOTATION

a = radius of fluid sphere
A_1, A_2, A_3 = coefficients of external stream function
B_1, B_2, B_3 = coefficients of external stream function
C_1, C_2, C_3 = coefficients of internal stream function
C = drag coefficient
C_{form} = form drag coefficient
C_{friction} = friction drag coefficient
D_1, D_2, D_3 = coefficients of internal stream function
D^2 = defined as $\dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right]$
D^4 = defined as $\left[\dfrac{\partial^2}{\partial r^2} + \dfrac{\sin\theta}{r^2}\dfrac{\partial}{\partial\theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\right]\right]^2$
$f_1(r,\theta)$ = defined by Equation (6)
$f_2(r,\theta)$ = defined by Equation (7)
f = body force vector
$g_0(r,Z)$ = defined by Equation (27)
$g_1(r,Z)$ = defined by Equation (34)
i_k = quantities appearing in Equation (42)
j_k = quantities appearing in Equation (43)
k = consistency index of the power law fluid
l, m = defined by Equation (47)
n = flow behavior index
$N_{Re_i} = \dfrac{aV_\infty\rho}{\mu_i}$, internal Reynolds number
$N_{Re_o} = \dfrac{\rho_0 V_\infty^{2-n} a^b}{K}$, external Reynolds number
p = pressure
r = dimensionless coordinate given as R/a
R = radial distance from the center of the sphere
s, t = defined by Equation (45)
V = relative velocity between fluid sphere field fluid
v = dimensionless velocity vector in Equations (2) and (3)
v_r, v_θ = dimensionless velocity component, related to the stream function as $-\dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta}$ and $\dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial r}$, respectively
V_∞ = physical velocity component
$X = \dfrac{\mu_i}{K}\left(\dfrac{a}{V_\infty}\right)^{n-1}$
Z = $\cos\theta$

Greek Letters

β_i = quantities appearing in Equation (22)
$\overline{\beta}_i, \overline{\beta}_i'$ = quantities appearing in Equation (35)
$\gamma_{i,j}$ = defined by Equations (40*b*) and (41)
$\delta_{i,j}$ = defined by Equations (40*c*) and (41)
Δ_j^i = rate of deformation tensor, defined as $V^i{}_{,j} + V^i{}_{,i}$
μ = viscosity
ψ = dimensionless stream function; ψ_i = interior fluid; ψ_o = exterior fluid
ϕ = defined by Equation (8)
ϕ_0 = defined by Equation (24)
ϕ_1 = defined by Equation (30)
ϕ_2 = $2(\phi_1 - \phi_0)$
ρ = density
λ_i = defined by Equation (9)
$\tau_{i,j}$ = stress tensor
$\tau^+{}_{i,j} = \dfrac{a^n}{KV_\infty{}^n\phi}\tau_{i,j}$ = dimensionless stress tensor
θ = angle

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